Arithmetic statistics of rational matrices of bounded height

Muhammad (Afif) Afifurrahman
joint work with Alina Ostafe and Igor E. Shparlinski

School of Mathematics and Statistics
University of New South Wales, Sydney, Australia

5 December 2023

Motivation: the integer matrices

- Consider a set $\mathcal{M}$ of matrices. We would like to count the number of matrices in $\mathcal{M}$ with a given rank, determinant, or characteristic polynomial.
- There have been a lot of works on the statistics of matrices in $\mathcal{M}_n(\mathbb{Z}; H)$, the set of $n \times n$ integer matrices with entries bounded by H in absolute value.
- Katznelson (1994) gave an asymptotic formula on the number of matrices in $\mathcal{M}_n(\mathbb{Z}; H)$ with a given rank.
- Katznelson (1993), Duke-Rudnick-Sarnak (1994), and Shparlinski (2010) gave bounds on the number of matrices in $\mathcal{M}_n(\mathbb{Z}; H)$ with a given determinant.
- Ostafe and Shparlinski (2022) bounded the number of matrices in $\mathcal{M}_n(\mathbb{Z}; H)$ with a given characteristic polynomial.
- We also have similar results on matrices over finite fields.

From integer matrices to rational matrices

- A natural extension of this family of problems is to replace integer matrices with rational matrices (of restricted height).
- We consider two different sets of rational numbers in our work,

$$\mathcal{F}(H) = \{a/b : \ a, b \in \mathbb{Z}, \ 0 \le |a|, b \le H, \ \gcd(a, b) = 1\},$$
$$\mathcal{E}(H) = \{1/a : \ a \in \mathbb{Z}, \ 1 \le |a| \le H\}.$$

  These are *Farey fractions* and *Egyptian/unit fractions* with height at most H.
- We note that their cardinalities satisfy

$$\#\mathcal{F}(H) \sim \frac{12}{\pi^2} H^2, \qquad \#\mathcal{E}(H) \sim 2H.$$

Working with rational matrices

- Based on these sets, define

$$\mathcal{M}_n(\mathbb{Q}; H) = \left\{ A = (a_{i,j})_{1 \le i,j \le n} \colon a_{i,j} \in \mathcal{F}(H), i,j = 1, \ldots, n \right\}$$

  as the set of $n \times n$ matrices whose entries are Farey fractions of height at most H.
- We also define $\mathcal{M}_n(\mathbb{Z}^{-1}; H)$ as the set of $n \times n$ matrices whose entries are Egyptian fractions of height at most H.
- We note that

$$\#\mathcal{M}_n(\mathbb{Q}; H) \sim \left(\frac{12}{\pi^2}\right)^{n^2} H^{2n^2}, \qquad \#\mathcal{M}_n(\mathbb{Z}^{-1}; H) \sim (2H)^{n^2}.$$

The problem over rational matrices

- We consider the problem of bounding the numbers of matrices in $\mathcal{M}_n(\mathbb{Q}; H)$ and $\mathcal{M}_n(\mathbb{Z}^{-1}; H)$ which have a given rank, determinant, or characteristic polynomial.
- Obstacle in working over rational numbers: Additions of two rational numbers of height H can result in a number of height H^2.
- Another obstacle: The sets $\mathcal{F}(H)$ and $\mathcal{E}(H)$ are not "discrete" and do not seem to yield to methods of geometry of numbers (e.g. counting over lattices).
- We are only concerned about the order of magnitude in our bounds.
- We use the notation

$$U \ll V \iff V \gg U \iff |U| \le cV$$

  for some positive constant c that only depends on the dimension n.
- We also write $U = V^{o(1)}$ if, for a fixed $\varepsilon > 0$, $V^{-\varepsilon} \le U \le V^{\varepsilon}$ for sufficiently big V.

Matrices with given rank

Let

$$L_{n,r}(\mathfrak{A}; H) = \#\left\{A \in \mathcal{M}_n(\mathfrak{A}; H) : \ \operatorname{rank} A = r\right\}.$$

with $\mathfrak{A} \in \{\mathbb{Q}, \mathbb{Z}^{-1}\}$. For the lower bound, we have

$$L_{n,r}(\mathbb{Q}; H) \gg H^{2nr}, \qquad L_{n,r}(\mathbb{Z}^{-1}; H) \gg H^{nr}$$

for matrices whose last $n - r$ rows are identical to the first row.

Matrices with given rank

For the case of integer matrices, Katznelson (1994) proved that $L_{n,r}(\mathbb{Z}; H)$ is asymptotically $cH^{nr} \log H$, for some $c > 0$ not depending on H.

MA, Ostafe, Shparlinski (2023+)

For all $n \geq 2$ and $r \geq 1$,

$$L_{n,r}(\mathbb{Q}; H) \leq H^{r(3n-r-1)+n+o(1)},$$

$$L_{n,r}(\mathbb{Z}^{-1}; H) \leq \begin{cases} H^{n+o(1)}, & r = 1, \\ H^{3n-2+o(1)}, & r = 2, \\ H^{(n-r)(r+1)/2+rn+o(1)}, & r \geq 3. \end{cases}$$

We also have a version of this bound for $m \times n$ matrices.

Sketch of the proof: given rank, $L_{n,r}(\mathfrak{A}, H)$

Consider a matrix A with entries in $\mathcal{F}(H)$ and rank r:

$$\left(\begin{array}{c|c} A_r & C_1 \\ \hline B_r & C_2 \end{array}\right).$$

- We fix an invertible $r \times r$ matrix A_r in H^{2r^2} ways.
- Key observation: each of the other rows of A can be represented as a unique linear combination of the first r rows of A.
- We count the possible number of choices of the $(n-r) \times r$ matrix B_r and $r \times (n-r)$ matrix C_1 based on the number t of nonzero coefficients of the corresponding linear combination.
- We will have a unique choice for the rest of the entries.

Sketch of the proof: given rank, $L_{n,r}(\mathfrak{A}, H)$

- In particular, for bounding the number of choices for C_1, we need to bound the number of solutions of equations of the form

$$\rho_1(h)a_{1,j} + \ldots + \rho_r(h)a_{r,j} - a_{h,j} = 0,$$

with $t+1$ nonzero coefficients $\rho_i(h)$ and $a_{1,j}, \ldots, a_{r,j}, a_{h,j} \in \mathcal{F}(H)$ for some indices h and j.

- We eventually have

$$L_{n,r}(\mathbb{Q}; H) \leq H^{2r^2} \sum_{t=1}^{r} H^{2t(n-r)} (H^{2r-t+1+o(1)})^{n-r} \leq H^{r(3n-r-1)+n+o(1)}.$$

- We apply similar arguments to bound $L_{n,r}(\mathbb{Z}^{-1}; H)$.

Matrices with given determinant

Let

$$\mathcal{D}_n(\mathfrak{A}; H, \delta) = \{A \in \mathcal{M}_n(\mathfrak{A}; H) : \ \det A = \delta\}.$$

For the lower bound, we have

$$\#\mathcal{D}_n(\mathbb{Q}; H, 0) \gg H^{2n^2-2n}, \qquad \#\mathcal{D}_n(\mathbb{Z}^{-1}; H, 0) \gg H^{n^2-n}, \qquad \#\mathcal{D}_n(\mathbb{Q}; H, 1) \gg H^{n^2+o(1)},$$

attained by matrices of the form

$$\begin{pmatrix} a_{1,1} & \dots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n-1,1} & \dots & a_{n-1,n} \\ a_{n-1,1} & \dots & a_{n-1,n} \end{pmatrix}, \qquad \begin{pmatrix} p_1/p_2 & a_{1,2} & \dots & a_{1.n-1} & a_{1,n} \\ 0 & p_2/p_3 & \dots & a_{2,n} & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & p_{n-1}/p_n & a_{n-1,n} \\ 0 & 0 & \dots & 0 & p_n/p_1 \end{pmatrix}.$$

with p_i primes in $[1, H]$.

Matrices with given determinant

For integer matrices, we have, from Shparlinski (2010),

$$\#\mathcal{D}_n(\mathbb{Z}; H, \delta) \ll H^{n^2-n} \log H.$$

MA, Ostafe, Shparlinski (2023+)

For all $n \geq 2$ and $\delta \in \mathbb{Q}$,

$$\#\mathcal{D}_n(\mathbb{Q}; H, \delta) \leq \begin{cases} H^{4+o(1)}, & \text{if } n = 2, \\ H^{2n^2-n+o(1)}, & \text{if } n \geq 3. \end{cases}$$

$$\#\mathcal{D}_n(\mathbb{Z}^{-1}; H, \delta) \leq \begin{cases} H^{o(1)}, & \text{if } n = 2,\ \delta \neq 0, \\ H^{2+o(1)}, & \text{if } n = 2,\ \delta = 0, \\ H^{7+o(1)}, & \text{if } n = 3, \\ H^{n^2-n/2-1/(2n-2)+o(1)}, & \text{if } n \geq 4,\ \delta \neq 0, \\ H^{n^2-n/2+o(1)}, & \text{if } n \geq 4,\ \delta = 0. \end{cases}$$

Sketch of the proof: given determinant, $\#\mathcal{D}_n(\mathfrak{A}; H, \delta)$

For $n = 2$, we expand the equation directly. An example for $\#\mathcal{D}_2(\mathbb{Q}; H, \delta)$:

$$\begin{vmatrix} a_1/b_1 & a_2/b_2 \\ a_3/b_3 & a_4/b_4 \end{vmatrix} = r/s \iff rb_1b_2b_3b_4 + sa_2a_3b_1b_4 = sa_1a_4b_2b_3.$$

We then fix some elements in the last equation and use the divisor bound ($\tau(n) \ll n^{o(1)}$) to bound the number of choices for other variables.
For $n \geq 3$ and $\delta = 0$, we use the rank bound to get

$$\#\mathcal{D}_n(\mathfrak{A}; H, 0) \ll \sum_{r=0}^{n-1} L_{n,r}(\mathfrak{A}; H) \ll \begin{cases} H^{2n^2-n+o(1)}, & \text{if } \mathfrak{A} = \mathbb{Q}, \\ H^{n^2-n/2+o(1)}, & \text{if } \mathfrak{A} = \mathbb{Z}^{-1}. \end{cases}$$

Sketch of the proof: given determinant, $\#\mathcal{D}_n(\mathfrak{A}; H, \delta)$

When $\delta \neq 0$, we use Laplace expansion on the first row of a matrix $A \in \mathcal{D}_n(\mathfrak{A}; H, \delta)$ to get an equation of the form

$$\sum_{j=1}^{n} Q_j a_{1,j} = Q_0,$$

with $a_{1,j} \in \mathcal{F}(H)$ or $\mathcal{E}(H)$, for $j = 1, \ldots, n$. We then bound the number of solutions of this equation.

If $\mathfrak{A} = \mathbb{Q}$, we use a result of Shparlinski (2017) to show that this equation has at most $H^{n+o(1)}$ solutions in $\mathcal{F}(H)^n$. This implies

$$\#\mathcal{D}_n(\mathbb{Q}; H, \delta) \ll H^{2n^2-n+o(1)}.$$

A new result on the equation $\sum Q_i/x_i = Q_0$

If $\mathfrak{A} = \mathbb{Z}^{-1}$, the problem of bounding the number of solutions of the previous equation is equivalent to the following problem, to which we give a new result.

MA, Ostafe, Shparlinski (2023+)

Let $(Q_0, Q_1, \ldots, Q_n) \in \mathbb{Z}^{n+1}$ with $1 \le |Q_i| \le H^{O(1)}$ for $i = 1, \ldots, n$. Then, the equation

$$\sum_{i=1}^{n} Q_i/x_i = Q_0,$$

has at most $H^{n/2+o(1)}$ solutions $(1/x_1, \ldots, 1/x_n) \in \mathcal{E}(H)^n$. Furthermore, if $Q_0 \neq 0$, we may replace the exponent $n/2 + o(1)$ with $n/2 - 1/(2n-2) + o(1)$

The proof is based on bounding the number of integer solutions of this equation, with $|x_i| \le H$:

$$\mathrm{lcm}(x_1, \ldots, x_n) | Q x_1 \ldots x_n.$$

This result implies $\#\mathcal{D}_n(\mathbb{Z}^{-1}; H, \delta) \ll H^{n^2 - n/2 - 1/(2n-2) + o(1)}$ for $\delta \neq 0$

Matrices with given characteristic polynomial

Let $\mathcal{P}_n(\mathfrak{A}; H, f) = \{A \in \mathcal{M}_n(\mathfrak{A}; H) \colon$ the characteristic polynomial of A is $f\}$.
For integer matrices, we have, from Ostafe and Shparlinski (2022),

$$\#\mathcal{P}_n(\mathbb{Z}; H, f) \ll H^{n^2-n-1/(n-3)^2} \text{ if } n \geq 4.$$

If f splits in $\mathbb{Q}$, then $\#\mathcal{P}_n(\mathbb{Q}; H, f) \gg H^{n^2-n}$.

MA, Ostafe, Shparlinski (2023+)

$$\#\mathcal{P}_n(\mathbb{Q}; H, f) \ll \begin{cases} H^{3+o(1)}, & \text{if } n = 2, \\ H^{2n^2-2n}, & \text{if } n \geq 3. \end{cases}$$

$$\#\mathcal{P}_n(\mathbb{Z}^{-1}; H, f) \ll \begin{cases} H^{o(1)}, & \text{if } n = 2 \text{ and } f(X) \neq X^2, \\ H^{n^2-n}, & \text{if } n \geq 3. \end{cases}$$

$$\#\mathcal{P}_2(\mathbb{Z}^{-1}; H, X^2) = \frac{24}{\pi^2} H \log^2 H + O(H \log H).$$

Thank you