Multiplicatively dependent integer vectors on a hyperplane

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Pictured: The three co-authors.

Multiplicatively dependent vectors

A vector $\boldsymbol{\nu} \in (\mathbb{C}^\times)^n$ is *multiplicatively dependent* if there exists a nonzero vector $\mathbf{k} \in \mathbb{Z}^n$ with

$$\nu_1^{k_1} \dots \nu_n^{k_n} = 1.$$

Convention: Following previous works, we do not allow $\nu_i = 0$ in this work. However, the results are extendable by letting $0^0 = 1$. **Examples**:

$$\begin{aligned} &(1, \dots, \nu_n) && [1^1 \dots \nu_n^0 = 1], \\ &(x, x, \dots, \nu_n) && [x^1 x^{-1} \dots \nu_n^0 = 1], \\ &(x, -x, \dots, \nu_n) && [x^2 (-x)^{-2} \dots \nu_n^0 = 1]. \end{aligned}$$

The set of multiplicatively dependent vectors forms an algebraic subgroup of $\mathbb{G}_m^n$.

Previous works

There have been lots of works regarding the vector $\boldsymbol{\nu}$ or the exponent $\mathbf{k}$. Two examples:

- Pappalardi-Sha-Shparlinski-Stewart (2018): asymptotical formula for the number of multiplicatively dependent vectors of algebraic integers of a fixed degree or in a number field and of a bounded height H.
- Bombieri-Masser-Zannier (1999-2008): “unlikely intersections” of a variety (curve or planes) with algebraic subgroups of high codimension.

Goals of this work: asymptotical formulae for the number of multiplicatively dependent vectors of bounded height that lies on a fixed hyperplane.

The main object for this talk is multiplicatively dependent vectors $\boldsymbol{\nu} \in \mathbb{Z}^n$ of height H (inside a box $[-H, H]^n$) which lies on a hyperplane

$$\Gamma : \; \alpha_1 x_1 + \cdots + \alpha_n x_n = J,$$

for a fixed $\boldsymbol{\alpha} \in \mathbb{Z}^n$ and an integer J. We are interested in finding an asymptotical formula for the number of such vectors, denoted as $S_n(H, J; \boldsymbol{\alpha})$, with $H \to \infty$.

Recalling PSSS

Pappalardi-Sha-Shparlinski-Stewart proved the number of multiplicatively dependent integer vectors $\nu \in \mathbb{Z}^n$ in the box $[-H, H]^n$ as $H \to \infty$ is

$$2^{n-1} n(n+1) H^{n-1} + O(H^{n-2} \exp(c_n \log H / \log\log H)).$$

On the other hand, the number of vectors $\nu \in \mathbb{Z}^n$ in the box $[-H, H]^n$ on the hyperplane Γ is of order H^{n-1}.
Thus, we expect the number of vectors $\mathcal{S}_n(H, J; \boldsymbol{\alpha})$ in our setup is of order H^{n-2}. We confirm this heuristic in the next slide.

The main result

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Suppose $n \geq 5$, $J \in \mathbb{Z}$ and $\boldsymbol{\alpha} \in \mathbb{Z}^n$ with $k \geq 2$ nonzero elements. Then, as $H \to \infty$, there exists a computable constant $C_{\boldsymbol{\alpha};J} \geq 0$ with

$$S_n(H, J; \boldsymbol{\alpha}) = C_{\boldsymbol{\alpha};J} H^{n-2} + \begin{cases} O(H^{n-5/2}) & \text{if} \quad k \geq 5, \\ O(H^{n-5/2}(\log H)^{16}) & \text{if} \quad k = 2, 3, 4 \text{ and } J \neq 0. \end{cases}$$

The leading coefficient $C_{\boldsymbol{\alpha}_J}$ and the implied constant in the error term depend on $\boldsymbol{\alpha}$ and J.

Sketches of proof: Multiplicative rank of a vector

We divide the counting based on the *(multiplicative) rank* of a vector $\boldsymbol{\nu}$, the largest number r such that any r-multiset of the coordinates of $\boldsymbol{\nu}$ is multiplicatively independent. If $\boldsymbol{\nu}$ has a coordinate ± 1, we let the rank be zero. For examples,

- $(1, \nu_2, \dots)$ is of rank 0,
- $(2, 2, \dots)$, $(2, -4, \dots)$ is of rank ≤ 1,
- $(2, 3, 6, \dots)$, $(2, 3, -12, \dots)$ is of rank ≤ 2.

Let the number of vectors with rank r be $S_{n,r}(H, J; \boldsymbol{\alpha})$, then

$$S_n(H, J; \boldsymbol{\alpha}) = S_{n,0}(H, J; \boldsymbol{\alpha}) + \cdots + S_{n,n-1}(H, J; \boldsymbol{\alpha}).$$

Our argument to count $S_n(H, J; \boldsymbol{\alpha})$ consists of two main parts:

- proving there are "a few" vectors of rank between 2 and $n - 1$ and
- counting vectors of rank 0 and 1.

Vectors with large rank

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Let n be a positive integer, and $\boldsymbol{\alpha} \in \mathbb{Z}^n$ with k nonzero coordinates. Then, for all nonnegative integers $r < n$, there exists $c, c_r > 0$ with

$$S_{n,r}(H, J; \boldsymbol{\alpha}) < \begin{cases} c_r H^{n-1-\lceil (r+1)/2 \rceil} \exp(c \log H / \log\log H) & \text{if} \quad r \leq k-2, \\ c_r H^{n-\lceil (r+1)/2 \rceil} \exp(c \log H / \log\log H) & \text{otherwise,} \end{cases}$$

Main idea: For a fixed $\mathbf{k}$, consider the related hyperplane and multiplicative equations:

$$\alpha_{i_1}\nu_{i_1} + \cdots + \alpha_{i_k}\nu_{i_k} = J,$$
$$\nu_{j_1}^{k_{j_1}} \ldots \nu_{j_s}^{k_{j_s}} = \nu_{j_{s+1}}^{k_{j_{r+1}}} \ldots \nu_{j_{r+1}}^{k_{j_{r+1}}}.$$

Fix the first equation in $O(H^{k-1})$ ways, then fix the second equation in $O(H^{\max(s,r+1-s)+o(1)})$ ways (for a fixed $\mathbf{k}$), then fix the rest of the variables in H^{n-k-r} ways.
However, overlaps between the indices $\mathbf{i}$ and $\mathbf{j}$ may happen, which give the condition $r \leq k-2$.

Large k, small k

The previous lemma implies

$$S_n(H,J;\boldsymbol{\alpha}) = \begin{cases} S_{n,0}(H,J;\boldsymbol{\alpha}) + S_{n,1}(H,J;\boldsymbol{\alpha}) + O(H^{n-3+o(1)}), & \text{when } k \geq 5 \\ \sum_{r=0}^{3} S_{n,r}(H,J;\boldsymbol{\alpha}) + O(H^{n-3+o(1)}), & \text{else.} \end{cases}.$$

It remains to improve the upper bound for $k \leq 4$ and $r = 2,3$. This corresponds to counting integer solutions (when, for example $k = 3$, $r = 2$) to this system of equations for a fixed $\mathbf{k}$:

$$\alpha_1\nu_1 + \alpha_2\nu_2 + \alpha_3\nu_3 = J,$$
$$\nu_1^{k_1}\nu_2^{k_2} = \nu_3^{k_3},$$

with $|\nu_1|, |\nu_2|, |\nu_3| \leq H$.

Hyperplane and multiplicative equation

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Let A, B, α_1, α_2, α_3, J be nonzero integers and k_1, k_2, k_3 be positive integers. Then, the number of integer solutions (ν_1, ν_2, ν_3) to the system of equations

$$\begin{aligned} \alpha_1\nu_1 + \alpha_2\nu_2 + \alpha_3\nu_3 &= J, \\ A\nu_1^{k_1}\nu_2^{k_2} &= B\nu_3^{k_3}. \end{aligned}$$

such that $\alpha_1\nu_1, \alpha_2\nu_2 \neq J$ and $0 < |\nu_i| \leq H$ for $i = 1, 2, 3$ is bounded above by

$$C_2(k_1 + k_2 + k_3)H^{1/2}(\log H + 2) + C_3(k_1 + k_2 + k_3)^3 H^{1/3}(\log H + k_1 + k_2 + k_3)$$

for some absolute constants $C_2, C_3 > 0$.

Integer points on curves

By substitution, we obtain (α_1, α_2) is an integer point of naive height at most H on the curve

$$f(x,y) = A\alpha_3^{k_3} x^{k_1} y^{k_2} - B(J - \alpha_1 x - \alpha_2 y)^{k_3}.$$

We may use Bombieri-Pila's determinant method to count the number of such points.
In particular, Castryck-Cluckers-Dittmann-Nguyen (2020) proved for any integral affine curve $g \subseteq \mathbb{A}^2_{\mathbb{Q}}$ of degree d, there exists an absolute constant $c > 0$ such that for all $H \geq 1$, the number of integer points in g with naive height at most H is at most $cd^3 H^{1/d}(\log H + d)$.
However, we do not know whether the curve f is irreducible!

Modifying Bombieri-Pila

Fortunately, we can apply the Bombieri-Pila method to a curve g of degree d with the following strategy:

- Suppose $g = g_1 \dots g_n$, where g_i is irreducible.
- Prove no integer points of g lies on g_i when $\deg g_i = 1$.
- For other g_i with $\deg g_i \geq 2$, apply Bombieri-Pila to these curves to obtain at most

$$c(\deg g_i)^3 H^{1/\deg g_i}(\log H + \deg g_i)$$

  integer points.
- Adding over all g_i, we obtain the following result (in the next slide) for a general g:

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Let $g \subseteq \mathbb{A}^2_{\mathbb{Q}}$ be an affine curve of degree d such that any linear factor of g does not have any integer points. Then there exist absolute constants $C_0, C_1 > 0$ such that for all $H \geq 1$, the number of integer points on g with naive height at most H is at most

$$C_0 d H^{1/2}(\log H + 2) + C_1 d^3 H^{1/3}(\log H + d).$$

After obtaining this result, we return to our curve

$$f(x, y) = A\alpha_3^{k_3} x^{k_1} y^{k_2} - B(J - \alpha_1 x - \alpha_2 y)^{k_3},$$

for a fixed $\mathbf{k}$. The only possible linear factors of f are of the form $x - Dy - E$, $x - E$ and $y - E$. We prove that each of these cannot have any integer points of f, and apply our quantitative result. Then, we repeat these arguments for other values of r and k.

Returning to the original problem

Concluding the arguments, when $J \neq 0$ and $2 \leq k \leq 4$, we have

$$S_{n,2}(H, J; \boldsymbol{\alpha}) + S_{n,3}(H, J; \boldsymbol{\alpha}) = O(H^{n-5/2}(\log H)^{16}),$$

which implies

$$S_n(H, J; \boldsymbol{\alpha}) = S_{n,0}(H, J; \boldsymbol{\alpha}) + S_{n,1}(H, J; \boldsymbol{\alpha}) + \begin{cases} O(H^{n-3+o(1)}), & \text{when } k \geq 5, \\ O(H^{n-5/2}(\log H)^{16}), & \text{when } 2 \leq k \leq 4 \text{ and } J \neq 0. \end{cases}$$

Thus, it remains to count the number of multiplicatively dependent integer vectors of bounded height H with rank 0 and 1. For demonstrations, we let $\boldsymbol{\alpha} = (1, 2, \ldots, n)$, with $n \geq k \geq 5$.

Counting vectors of rank 0

When $\boldsymbol{\nu}$ is of rank 0, we need to count the number of vectors $\boldsymbol{\nu} \in \mathbb{Z}^n$ of height at most H such that there exists an i with $\nu_i = \pm 1$. If $\boldsymbol{\nu}_1 = 1$, we need to count the number of vectors $(\nu_2, \ldots, \nu_n) \in \mathbb{Z}^{n-1}$ inside the box $[-H, H]^{n-1}$ that lie on the hyperplane

$$2\nu_2 + \cdots + n\nu_n = J - 1.$$

Davenport's lemma allows us to translate this point-counting problem to computing volume of a section of the hyperplane $\alpha^* \cdot \boldsymbol{\nu} = 0$ in the box $[-1/2, 1/2]^{n-1}$. Then, such volume is computed using Marichal-Mossinghoff (2006).

Therefore, the number of such vectors $\boldsymbol{\nu}$ is

$$\frac{2^{n-2}}{\sqrt{2^2+\cdots+n^2}}\mathrm{Vol}_{n-1}\left(\{\boldsymbol{\nu}\in[-1/2,1/2]^n : 2\nu_2+\cdots+n\nu_n=0\}\right)H^{n-2}+O(H^{n-3}).$$

We repeat this argument for each of the n coordinates, each corresponds to a dfferent new hyperplane. We also consider the case $\nu_i=-1$.
Adding all terms, excluding "double cases" and the case $\nu_i=0$, we obtain

$$S_{n,0}(H,J;\boldsymbol{\alpha})=\mathcal{C}_0H^{n-2}+O(H^{n-3}),.$$

Counting vectors of rank 1

For a vector $\boldsymbol{\nu}$ of rank 1, there exist two coordinates ν_{i_1}, ν_{i_2} and positive integers k_1, k_2 with

$$\nu_{i_1}^{k_1} = \nu_{i_2}^{k_2}.$$

By bounding the exponent, if $k_1 \neq k_2$, there are at most $O(H^{n-5/2})$ vectors in $\mathbb{Z}^n$.
Then, when $k_1 = k_2$, we have $\nu_{i_1} = \pm\nu_{i_2}$.For example, when $i_1 = 1, i_2 = 2$, we need to count the number of integer vectors $\boldsymbol{\nu}$ in the box $[-H, H]^{n-1}$ which lies on the hyperplane

$$3\nu_1 + 3\nu_3 + \cdots + n\nu_n = J.$$

We use similar arguments based on Davenport's lemma and Marichal-Mossinghoff to obtain the number of such vectors $\boldsymbol{\nu}$ in $[-H, H]^n$. We repeat these for the other pairs of indices and the case $\nu_{i_1} = -\nu_{i_2}$, add all terms and obtain

$$S_{n,1}(H, J; \boldsymbol{\alpha}) = \mathcal{C}_1 H^{n-2} + O(H^{n-5/2}).$$

Conclusion: the main result

Combining all ranks, we obtain the following.

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Suppose $n \geq 5$, $J \in \mathbb{Z}$ and $\boldsymbol{\alpha} \in \mathbb{Z}^n$ with $k \geq 2$ nonzero elements. Then, as $H \to \infty$, there exists a computable constant $C_{\boldsymbol{\alpha};J} \geq 0$ with

$$S_n(H, J; \boldsymbol{\alpha}) = C_{\boldsymbol{\alpha};J} H^{n-2} + \begin{cases} O(H^{n-5/2}) & \text{if} \quad k \geq 5, \\ O(H^{n-5/2}(\log H)^{16}) & \text{if} \quad k = 2, 3, 4 \text{ and } J \neq 0. \end{cases}$$

The $C_{\boldsymbol{\alpha}_J}$ and the implied constant in the error term depend on $\boldsymbol{\alpha}$ and J.

Other results

- For the case $k = 1$, the problem is counting multiplicatively dependent integer vectors with a fixed coordinate J. We take $\boldsymbol{\alpha} = \mathbf{e}_1 = (1, 0, \ldots, 0)$ and obtain the following.

MA-Iverson-Sanjaya (2025+)

Let $n \geq 3$ and $J \neq -1, 0, 1$ be an integer such that $|J|$ is not a perfect power. Then, there exists real constants $C_J^{(0)}$, $C_J^{(1)}$ that depend on J with

$$S_n(H, J; \mathbf{e}_1) = C_J^{(1)} H^{n-2} \left\lfloor \frac{\log H}{\log |J|} \right\rfloor + C_J^{(0)} H^{n-2} + O(H^{n-5/2}).$$

In this case, the main terms come from vectors of rank 0, 1 and 2.

- Our work actually obtain uniform formulae with respect to H and J when $k \geq 3$. In addition, we also worked on similar problems when we restrict $\boldsymbol{\nu}$ to have positive coordinates. Most arguments presented here work in these setups, with some technicalities and different terms.

Further remarks and extensions

- For virtually all choices for α and J, the constant $C_{\alpha;J}$ in the main result is nonzero. However, there exists a choice of α and J such that this constant is zero, based on some system of linear equations.
- The $O(H^{n-5/2})$ error term comes from vectors of the form $(x, x^2, \ldots)$, thus this error term is strong. However, the $(\log H)^{16}$ term is not expected.
- Many parts of our arguments still work if we replace integers with algebraic integers or numbers of height H and of fixed degree or in a fixed number field, as in Pappalardi-Sha-Shparlinski-Stewart's work. However, arguments from Bombieri-Pila or Marichal-Mossinghoff are not readily available in this setup.
- Another possible future directions: replacing hyperplane with other varieties, in particular quadratic forms. In this case, one may instead use Schwart-Zippel's lemma to bound the number of points.

Thank you

M. Afifurrahman, V. Iverson and G. C. Sanjaya, ‘Multiplicatively dependent integer vectors on a hyperplane’, Preprint, 2025.